Gauge theory RG flows from a warped resolved orbifold

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ABSTRACT: We study gauge-string duality of D3-branes localized at a point on the resolution of the ALE orbifold $\mathbb{C}^3/\mathbb{Z}_3$. By explicitly solving for the warp factor, we demonstrate the holographic RG flow from $AdS_5 \times S^5/\mathbb{Z}_3$ (far away from the resolution) to the usual $AdS_5 \times S^5$ (close to the stack). On the gauge theory side, this maps to the flow between the quiver gauge theory and $\mathcal{N} = 4$ super Yang-Mills. We present two possible scenarios for this RG flow depending on the choice of the VEVs. In particular, one of the scenarios proceeds by two steps involving both Higgsing and confinement.

KEYWORDS: AdS-CFT Correspondence, D-branes, Supersymmetric gauge theory.

Contents

1. Introduction

One powerful way of constructing gauge theories with reduced supersymmetry in string theory is to consider stacks of D-branes at Calabi-Yau singularities [1]. The original proposal of the AdS/CFT correspondence [2] compared string theory on the near-horizon region of a stack of D3-branes in flat space to the low energy $\mathcal{N} = 4$ super Yang-Mills theory living on the branes. The zoom-in involved in the near-horizon limit washes out all the details of the original background, so changing the background geometry from flat to curved does not result in new gauge theories. One way to get past this problem and to construct new gauge theories, is to look at D3-branes not at a smooth point, but instead at the tip of a Calabi-Yau cone. If the cone Y_6 is a 6-dimensional real cone over a compact base space X_5,

then the near-horizon limit gives rise to string theory on[1] $AdS_5 \times X_5$ and the dual gauge theory is generically a quiver theory whose details depend on the nature of the singularity.

To construct phenomenologically interesting theories, we need less supersymmetry and we also need to break conformal invariance. One way to break some of the SUSY is to replace the S^5 in $AdS_5 \times S^5$ with an orbifold S^5/Γ, where Γ is a discrete group. This amounts to considering $\mathbb{R}^{3,1} \times \mathbb{R}^6/\Gamma$ instead of $\mathbb{R}^{9,1} = \mathbb{R}^{3,1} \times \mathbb{R}^6$ before the near-horizon limit. If $\Gamma \in \text{SU}(2)$, then the dual theory is $\mathcal{N} = 2$, and if $\Gamma \notin \text{SU}(2)$ then we have an $\mathcal{N} = 1$ gauge theory.[2] In this paper, we will consider the case $\Gamma = \mathbb{Z}_3$ and the dual $\mathcal{N} = 1$ superconformal quiver. Another class of examples with reduced supersymmetry where the explicit Ricci-flat metric on the cone is known, is when the base X_5 is topologically $S^2 \times S^3$. The most extensively investigated gauge-theory/singular-geometry dual of this kind is where Y_6 is chosen to be the conifold [9], so that the base space is a coset space called $T^{1,1}$. More general classes of Ricci-flat metrics on cones over $S^2 \times S^3$ are also known, leading to the so-called $Y^{p,q}$ and (even more general) $L^{a,b,c}$ spaces and their $\mathcal{N} = 1$ gauge theory duals [16, 21, 29, 30, 40 – 55].

In [1, 3], the dual $\mathcal{N} = 1$ superconformal gauge theory corresponding to a stack of N D3-branes at the tip of the conifold was argued to be an $\text{SU}(N) \times \text{SU}(N)$ gauge theory with matter content given by two chiral superfields A_i and B_i (where $i = 1, 2$). The fields A_i transform in $(N, \bar{N})$ of $\text{SU}(N) \times \text{SU}(N)$ while B_i transform in $(\bar{N}, N)$. The gauge theory on the singular conifold corresponds to the case where none of the fields have a VEV, and the theory is conformal.

To break conformal invariance, we need to look at a space that is not a cone, because the radial direction of the cone is what gets absorbed into the AdS^5 when we take the near-horizon limit.[3] In the case of the conifold theory described above, there are two ways in which we can accomplish this. One is to *deform* it by considering a non-vanishing S^3 at the tip, and the other is to *resolve* it and keep the S^2. The former case was the subject of the celebrated paper of Klebanov and Strassler [32] where the deformed S^3 is supported by a non-vanishing 3-form flux. The dual theory was found to be a cascading gauge theory (related, but different, from the gauge theory on the singular space). The resolved conifold has also been investigated, but in the dual gauge theory, it corresponds to giving a non-zero VEV for the operator

$$\mathcal{U} = \frac{1}{N}\text{Tr}\left(\left|A_1^2\right| + \left|A_2^2\right| - \left|B_1^2\right| - \left|B_2^2\right|\right) \tag{1.1}$$

in the *same* $\text{SU}(N) \times \text{SU}(N)$ gauge theory encountered in the case of the singular conifold. The dual supergravity solution under the assumption that the branes are smeared on the S^2 was constructed in [13]. The solution they found was singular, as expected. Ideally though, one would like to see the RG flow created by the specific non-zero VEVs on the supergravity side as well, and see the flow from $AdS_5 \times T^{1,1}$ in the UV to $AdS^5 \times S^5$

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[1]The way in which this happens is easily seen from, e.g., the discussion we give in section 3.1.

[2]We still need $\Gamma \in \text{SU}(3)$, to satisfy the Calabi-Yau condition.

[3]The global symmetry $\text{SO}(4, 2)$ of AdS^5 is what gets translated to the conformal symmetry of the gauge theory.

near the stack, when the stack is localized on a specific point on the resolution. This was accomplished in [12], where they found the precise form of the warp factor.

In this paper we consider instead the space $\mathbb{C}^3/\mathbb{Z}^3$, which after the near-horizon limit gives rise to $AdS_5 \times S^5/\mathbb{Z}_3$. We look at the case where the space gets a resolution, and the orbifold singularity is replaced by a four-cycle instead of the two-cycle of the conifold. This four-cycle is the projective space $\mathbb{CP}^2$. We consider the case where the stack of D-branes is localized on a point on the resolution. Far away from the resolution, we see that the warp factor tends to the unresolved case while close to the stack we see the emergence of the AdS throat because the stack is no longer at a singular point. Exactly like in the conifold case there is a single parameter in the resolved metric, which controls the size of the "blown-up" cycle. In the dual gauge theory, this results in a non-zero VEV for a six-dimensional, yet to be constructed [39] operator. On the gauge theory side it corresponds to the fact that all baryonic symmetries are anomalous and no simple current of the form (1.1) is preserved. It means, that unlike in the conifold case there is no straight-forward way to figure out what fields acquire VEVs. However, using the relation between the holomorphic coordinates on the orbifold and the chiral superfields of the gauge theory we found that there are only two non-identical patterns for the RG flow triggered by different VEVs of the fields. In both cases the VEVs breaks the conformal invariance and launches an RG flow that yields the $\mathcal{N} = 4$ SYM, but one of the two scenarios is more complicated then the other and involves two steps involving Higgsing and confinement, the details of which are presented in section 5.

The structure of this paper is as follows. In the next section we explain the geometry of the orbifold and its resolution. In section 3 we explain the supergravity solution corresponding to D3-branes on the unresolved space, and the corresponding dual quiver gauge theory. The next two sections are dedicated to the construction of the solution in the resolved space and a discussion of the RG flow on both the gravity and gauge theory sides of the duality. We end in the final section with some discussions and comments. Some of the technical details can be found in the appendices.

Various other aspects of the $\mathbb{C}^3/\mathbb{Z}_3$ orbifold have been studied in [10, 15, 7, 33]. Other recent papers which are of relevance to our work are [34 – 38].

2. The toric description of $\mathbb{C}^3/\mathbb{Z}_3$ and its resolution

The space $\mathbb{C}^3/\mathbb{Z}_3$ is defined as the three-dimensional complex space $\mathbb{C}^3$ under the identification

$$\{w_1, w_2, w_3\} = \{\eta w_1, \eta w_2, \eta w_3\}, \text{ with } \eta^3 = 1. \tag{2.1}$$

Clearly, the only orbifold point is the origin. The orbifold inherits the metric

$$ds_6^2 = dr^2 + r^2 d\Omega_{S^5/\mathbb{Z}_3}^2 \tag{2.2}$$

from the flat metric on $\mathbb{C}^3$. The subscript on the angular part is supposed to indicate that the periodicities of the angles are of the orbifold, not of flat space.

There is another, more algebraic, way in which we could define the above orbifold. This turns out to be very useful for comparisons with the gauge theory, so we explain it

now. The idea is that the orbifold is fully defined by the invariant (under the orbifold action) monomials that one can construct from the coordinates w_i. There are ten such monomials as is easily checked, and the domain over which these collection of monomials is well-defined is precisely $\mathbb{C}^3/\mathbb{Z}_3$. So one could also define our orbifold by

$$\mathbb{C}^3/\mathbb{Z}_3 = \mathbb{C}[w_1^3,\, w_1^2 w_2,\, w_1^2 w_3,\, w_2^3,\, w_2^2 w_1, w_2^2 w_3,\, w_3^3,\, w_3^2 w_1,\, w_3^2 w_2,\, w_1 w_2 w_3]. \qquad (2.3)$$

If we call these monomials U_{ijk}, with indices $i, j, k = 1, 2, 3$, then the algebraic relations between U_{ijk} (e.g., $U_{112}U_{123} = U_{223}U_{111}$) captures the geometry of the orbifold. It should be understood that U_{ijk} with different permutations of i, j, k are identified. So we could also write,

$$\mathbb{C}^3/\mathbb{Z}_3 = \mathbb{C}[U_{ijk}]/\{\text{algebraic relations between the } U_{ijk}\text{'s}\}. \qquad (2.4)$$

Another important useful fact about $\mathbb{C}_3/\mathbb{Z}_3$ is that it is toric. A quick and practical introduction to toric geometry can be found in appendix B of [29] or section 4 of [4]. Essentially, in the present context *toric* means that all the information about the space (as a complex variety) can be captured using a cone that has its apex at the origin of a 3-dimensional integer lattice. There are qualifications that need to be added to the cone to make this more precise, but instead of stating them, we will refer the exacting reader to [4–6]. In any event, it turns out[4] that the cone that defines our orbifold can be specified by the (rational) vertices that span it:

$$v_1 = (-1, 0, 1), \quad v_2 = (0, -1, 1), \quad v_3 = (1, 1, 1). \qquad (2.5)$$

One feature of toric spaces that are in addition Calabi-Yau is that their vertices (other that the origin) all have to lie on the same plane. Thus, we can capture them on the toric diagram presented in figure 1.

Toric diagrams are useful because many questions about the original space can be answered in terms of simple geometrical features of the diagram. The Calabi-Yau property being translated into co-planarity of vertices is a simple example. Another question that is readily answered is the question of singularities and their resolutions. In fact, the way in which the fact that $\mathbb{C}^3/\mathbb{Z}_3$ is singular is captured by our toric diagram is through the presence of a vertex, marked 4 in figure 1, in its interior. In general, if the volume of the cone with origin as its apex is greater than one, then the space is singular. Notice that the volume can be greater than one only[5] if there is an interior[6] point (which in our case is the vertex marked 4). This motivates the following simple recipe for resolving our singular space: add more vertices in the interior in such a way that the new cones have no vertices in *their* interior, and then look at the resulting collection of cones (the "fan") as the new

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[4]See appendix B.1 of [7] for a straightforward algorithm for reconstructing the space from its toric diagram.

[5]We are talking about triangulated toric diagrams. The toric diagram for the conifold, for example, is a square with no interior points, but still the variety is singular. There the resolution corresponds to triangulating the square, which amounts to blowing up a 2-sphere, not a 4-cycle.

[6]By "interior" here, we also mean points that are *on* a boundary. When the point is a genuine interior point like in our case, the resolution corresponds to the blowing-up of a four-cycle.
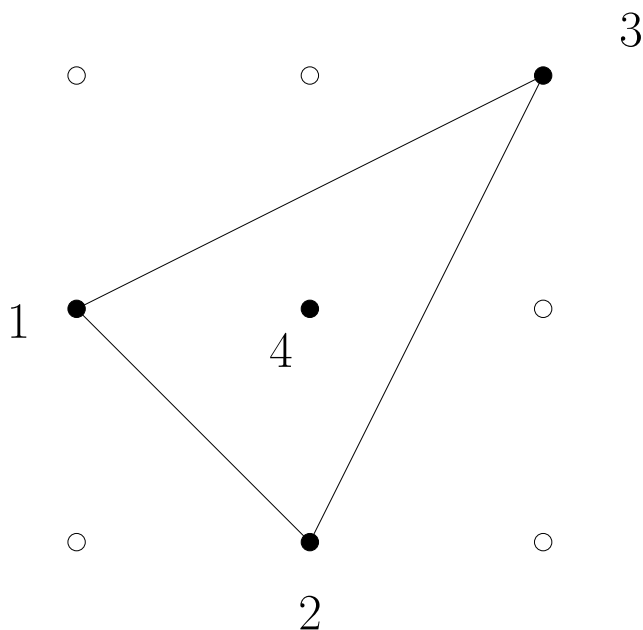
Figure 1: Toric Diagram of $\mathbb{C}^3/\mathbb{Z}_3$.

space. In our case, since there is only one interior point, $\mathbb{C}^3/\mathbb{Z}_3$ has a unique ("crepant") resolution, and the fan for the resolved space will involve the three new cones that have been created by the addition of vertex 4.

We will be interested in the resolution of the orbifold in later sections, so we make some comments about that here. The vector for vertex 4 is easily seen to be

$$v_4 = (0, 0, 1). \tag{2.6}$$

Clearly, the four vertices should satisfy 4 (= no. of vertices) - 3 (= dimensionality of the lattice) = 1 linear relation between them, which we write as

$$\sum_{i=1}^{4} Q_i v_i = 0, \text{ with } Q_i = (1, 1, 1, -3). \tag{2.7}$$

The reason why these charges Q_i are interesting is because the translation between the toric diagram and the actual space we are interested in, is effected through them. This is done by means of the "quotient construction" as follows. Pick 4 (= no. of vertices) complex coordinates z_i. We can define a $\mathbb{C}^*$-action on these coordinates

$$(z_1, z_2, z_3, z_4) \rightarrow (\lambda z_1, \lambda z_2, \lambda z_3, \lambda^{-3} z_4), \ \lambda \in \mathbb{C}^*, \tag{2.8}$$

where the powers are the charges above. At this stage, we are only defining the action, we have not yet modded out by anything. The space now is constructed by first imposing

$$|z_1|^2 + |z_2|^2 + |z_3|^2 - 3|z_4|^2 = \mu, \tag{2.9}$$

and then modding out by the phase part (the U(1)'s of (2.8)). This should be compared to the two-step construction of $\mathbb{CP}^1$ by first setting $|z_1|^2 + |z_2|^2 = \mu$ in $\mathbb{C}^2$, and then modding out by phases, instead of modding out by $\mathbb{C}^*$ in one go. Notice that the charges Q affect both steps of the quotienting operation (2.9)–(2.8). This quotient construction of the geometry finds its gauge theory analogue in the description of the moduli space in terms of D-terms and is the basis for Witten's Gauged Linear Sigma Model (GLSM) construction [8] applied to $\mathcal{N} = 1$ theories.

Now, we can try to see how this toric construction of (the resolution of) $\mathbb{C}^3/\mathbb{Z}_3$ ties up with the earlier definitions. First we notice that instead of doing the U(1) quotienting, we can parametrise the space through the zero-charge monomials defined by

$$z_1^3 z_4, \ z_1^2 z_2 z_4, \ z_1^2 z_3 z_4, \ z_2^3 z_4, \ z_2^2 z_1 z_4, z_2^2 z_3 z_4, \ z_3^3 z_4, \ z_3^2 z_1 z_4, \ z_3^2 z_2 z_4, \ z_1 z_2 z_3 z_4. \tag{2.10}$$

Obviously, these monomials satisfy the same algebra as the U_{ijk}'s in (2.3).

We claim that $\mu = 0$ corresponds to the unresolved case, and when $\mu > 0$ we get the resolved space where the origin is replaced by a $\mathbb{CP}^2$. For this, first note that when $z_4 = 0$, all the monomials above are zero, which corresponds to the orbifold point in the language of the U_{ijk}'s. But from (2.9) it is clear that when $\mu = 0$, $z_4 = 0$ forces $z_{1,2,3}$ to be zero and therefore we end up with a single point, but when $\mu > 0$, setting $z_4 = 0$ results in a $\mathbb{CP}^2$. Thus, μ can be thought of as the size of the $\mathbb{CP}^2$ on the resolved orbifold, and when it is zero we end up with $\mathbb{C}^3/\mathbb{Z}_3$. It is also possible to see the resolution using only the monomials U_{ijk} similar to the famous conifold construction of [9]. It follows from the algebra satisfied by U_{ijk}'s that *all* 3d complex vectors of the form

$$v_{jk}^T \equiv (U_{1jk}, U_{2jk}, U_{3jk}) \tag{2.11}$$

lie on the same 2d plane, which means that there exists a 3d vector λ we have

$$v_{jk}^T \cdot \lambda = 0 \qquad \text{for any} \quad j, k. \tag{2.12}$$

This fact follows directly from relations of the form $U_{1jk}U_{2j'k'} = U_{2jk}U_{1j'k'}$. The existence of λ becomes even more evident if we notice that $v_{jk}^T = z_j z_k (z_1, z_2, z_3)$ for arbitrary j and k. Now, the non-zero vector λ is fixed up to an overall re-scaling, which means that it defines a point on $\mathbb{CP}^2$. Furthermore, λ is well-defined everywhere except the origin, where *all* the vectors v_{jk} are identically zero. At the apex λ is however un-restricted and we resolve the orbifold by replacing the singular point with the λ-parametrized $\mathbb{CP}^2$. This is exactly the same $\mathbb{CP}^2$ we found setting $\mu > 0$.

Before we leave the geometry, we add some comments about how Higgsing in the gauge theory is described in the geometry. When $\mu > 0$ and $z_4 = 0$ the rest of the coordinates in (2.9) define $\mathbb{CP}^2$. Let us assume that $|z_3|$ is of the same order of magnitude as the parameter μ. It means that we consider the $z_3 \neq 0$ patch of the $\mathbb{CP}^2$. On this patch we can introduce the coordinates $u_1 \equiv z_1/z_3$ and $u_2 \equiv z_2/z_3$. Together with the monomial $U_{333} = z_3^3 z_4$ these coordinates parametrise the entire patch $z_3 \neq 0$. Indeed, starting from the set (u_1, u_2, U_{333}) we can easily determine the rest of the U_{ijk}'s. For example, we have $U_{123} = u_1 u_2 U_{333}$. If we now send $\mu \to \infty$ still focusing on the $z_3 \neq 0$ patch we arrive at the

regular $\mathbb{C}^3$ parametrized by u_1, u_2 and U_{333}. On the toric diagram it looks like we "chop" off the 3d node, while in the dual gauge theory it corresponds to the RG flow to the $\mathcal{N} = 4$ SYM theory.

3. Gauge-string duality on the unresolved orbifold

3.1 Supergravity with brane sources

When we put a stack of D3-branes in some background, they act as sources for the type IIB supergravity equations of motion. There is a standard ansätz for solving these equations of motion (see e.g. [11]), which is given by the following prescriptions for the various fields (the metric, the dilaton and the five-form):

$$ds^2 = H^{-1/2}(y)\eta_{\mu\nu}dx^\mu dx^\nu + H^{1/2}(y)ds_6^2, \tag{3.1}$$

$$\Phi = \text{const.}, \quad F_5 = (1 + *)dH^{-1} \wedge dx^0 \wedge \ldots \wedge dx^3, \tag{3.2}$$

The $*$ stands for the Hodge dual operator. The ds_6^2 piece in the metric denotes the dimensions transverse to the D3-branes, and is given in our case by (2.2). The worldvolume (3+1)-metric of the D-branes is Minkowskian, and the entire solution is essentially given by one function, $H(y)$, where y denotes the coordinates on the transverse space. This function is called the warp factor. With this ansätz, the supergravity EOMs (with source terms for the branes) reduce to a single equation, the Green's equation on the transverse space (y_0 denotes the location of the stack):

$$\Box_y H(y, y_0) = -\frac{C}{\sqrt{g_6}}\delta^6(y - y_0). \tag{3.3}$$

We denote the determinant of the 6-metric by g_6. The strength of the source is captured by $C = 2\kappa_{10}^2 T_3 N = (2\pi)^4 g_s N \alpha'^2$ where $T_3 = 1/(8\pi^3 g_s \alpha'^2)$ is the brane tension and $\kappa_{10} = 8\pi^{7/2} g_s \alpha'^2$ is the 10-D Newton's constant.

For the case of the unresolved $\mathbb{C}^3/\mathbb{Z}_3$, when we place the stack at the orbifold singularity, this equation can be immediately solved because the warp factor depends only on the radial coordinate. Green's equation takes the form

$$\frac{1}{r^5}\frac{\partial}{\partial r}\left(r^5 \frac{\partial}{\partial r}H\right) = -\frac{3C}{\pi^3 r^5}\delta(r). \tag{3.4}$$

The normalization of the delta function accounts for the fact that we are ignoring the angular dependence, it comes from an integral over the angles.[7] Away from the origin the equation is easily integrated, and integrating over the delta function fixes the constant of integration:

$$H(r) = \frac{L^4}{r^4} \quad \text{where} \quad L^4 = 12\pi g_s N \alpha'^2. \tag{3.5}$$

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[7] This is analogous to the fact that in 3-dimensions, if we are looking at sources at the origin ($r_0 = 0$), then the replacement $\frac{1}{r^2 \sin\theta}\delta(r - r_0)\delta(\theta - \theta_0)\delta(\phi - \phi_0) \to \frac{1}{4\pi r^2}\delta(r)$ is acceptable for test-functions that are sufficiently well-behaved at the origin. The 4π that arises is nothing but $\int_0^\pi \sin\theta d\theta \int_0^{2\pi} d\phi$.

With the usual substitution $z = L^2/r$, we end up with

$$ds^2 = \frac{L^2}{z^2}(dz^2 + \eta_{\mu\nu}dx^\mu dx^\nu) + L^2 d\Omega_{S^5/\mathbb{Z}_3} \qquad (3.6)$$

which is nothing but $AdS^5 \times S^5/\mathbb{Z}_3$ with equal radii $(= L)$ for both the AdS and the $S^5/\mathbb{Z}_3$.

Another thing that we could do is to put the stack away from the tip, in which case we expect that far away, the solution should still look like the one we found above. But close to the stack, now we should see the emergence of the AdS throat because the stack is now at a smooth point. The full solution will have a singularity at $r = 0$. We verify these expectations explicitly in an appendix.

The general picture is that when the stack is at the singularity we have the $AdS^5 \times S^5/\mathbb{Z}_3$ background, and this corresponds to zero VEVs for the fields in the dual gauge theory. When we resolve the space, that is equivalent to turning on specific VEVs and one of the purposes of this paper is to investigate the RG flows triggered by these VEVs.

3.2 The dual quiver theory

There are more or less standard recipes for constructing dual gauge theories corresponding to D-branes probing singularities. Building on the work of [19, 20], algorithms for constructing the dual gauge theory corresponding to generic toric singularities have been developed in a sequence of papers starting with [21]. The state of the art can be found in the excellent review by Kennaway [22].

But for Abelian orbifolds, like the one we are considering here, we can deduce the gauge theory from simple arguments starting with AdS/CFT duality for $AdS_5 \times S^5$ [23, 24]. It turns out that the gauge theory information can be captured using a simple quiver diagram (figure (2)). Quiver diagrams are merely a compact way of describing the field content in an $\mathcal{N} = 1$ theory. In our case, there are three gauge groups and the fields transform as bi-fundamentals. For example, A_i $(i = 1, 2, 3)$, transforms in the fundamental N of $U(N)_3$ and and the anti-fundamental $\bar{N}$ of $U(N)_2$ in the figure. Notice that we have three sets of three fields each, one set on each edge. We also assign charges for these fields under the $U(1)$ contained in the $U(N)$: fields in the fundamental of $U(N)$ are assigned a $+1$ under the associated $U(1)$ and those in the anti-fundamental are assigned a -1. For example, the A_i have a charge of -1 under $U(1)_2$.

Apart from the $U(1)_R$ symmetry and the $SU(3)$ symmetry that acts on the i indices, there are no anomaly-free global continuous symmetries in the theory. In particular, there are two candidates for the baryonic symmetry, but both prove to be anomalous as one can easily check. As we have already mentioned in the Introduction this fact is evident from the geometry, since the dual singular space has no two-cycle resolution, and so there is no parameter dual to the baryonic current. On the other hand, the theory enjoys some un-broken discrete symmetries. The precise form of these symmetries as well their action on various wrapped D-branes on the geometry side is beautifully explained in [10].

For all of the nodes one has $N_f = 3N_c$, where N_f denotes the number of the flavors and $N_c = N$ corresponds to the colour. Thus the β-functions vanish precisely and the theory is conformal.
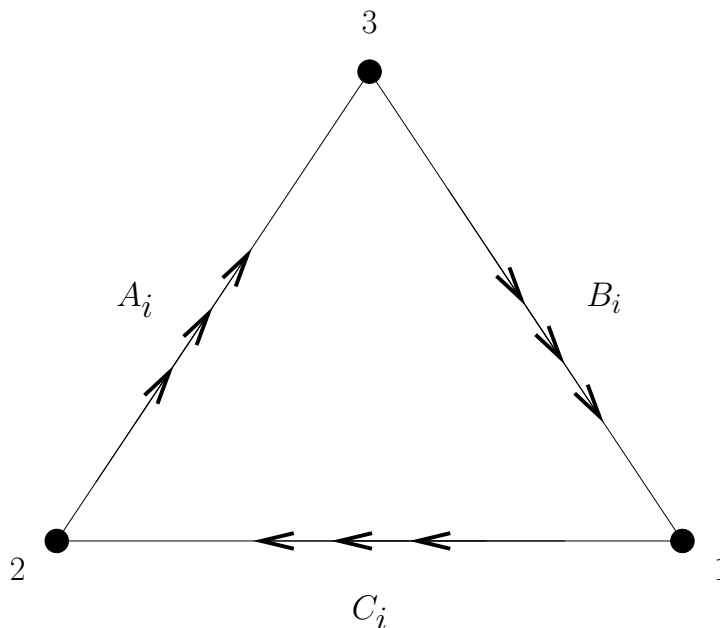
Figure 2: The quiver for the gauge theory dual to $\mathbb{C}^3/\mathbb{Z}_3$.

The quiver also determines the D-term equations that constrain the moduli space of vacua of the gauge theory. For the $\mathrm{U}(1)$ on the a-th node with Fayet-Iliopoulos parameter t_a, it takes the form $\sum_m Q_{a,m}|X_m|^2 = t_a$. The Q here are the charges, the X are the fields, and the summation is over all fields. For our case then,

$$\sum_i (|A_i|^2 - |B_i|^2) = t_3, \tag{3.7}$$

$$\sum_i (|B_i|^2 - |C_i|^2) = t_1, \tag{3.8}$$

$$\sum_i (|C_i|^2 - |A_i|^2) = t_2. \tag{3.9}$$

Here we assume for simplicty that the branes are moving together, which means that the fields A, B, C are assumed to be proportional to the identity. In the more general case, we will need to consider many copies of the space corresponding to independent motions of the branes. In any event, the fact that $t_1 + t_2 + t_3$ is identically zero, suggests that there are only two independent parameters instead of three: one corresponds to the resolution parameter a, and the other corresponds to the B field that can be turned on in the background [39].

The D-terms do not fully fix the vacuum moduli space as can be clearly seen by counting the degrees of freedom (DOF). There are 3×3 complex fields, and each D-term equation together with $\mathrm{U}(1)$ condition constrains one DOF each. But we also have to make allowance for the fact that the D-terms for all the gauge groups add up to zero from the equations above, so they are not all independent. Thus the vacuum moduli space according to the D-terms alone is $3 \times 3 - 3 + 1 = 7$ complex dimensional, in conflict with

our expectation that it should in fact be 3-dimensional. The resolution is of course that we haven't taken the F-terms into account yet, for which we need the superpotential.

The superpotential for the quiver is

$$W = \epsilon_{ijk}\text{Tr}\left(A_i B_j C_k\right). \tag{3.10}$$

The trace is over the gauge indices, and we will suppress writing it from now on. The F-term equations $\frac{\partial W}{\partial X} = 0$ (where X is A_i, B_i or C_i) therefore give rise to a bunch of algebraic relations of the form $B_2 C_3 = C_2 B_3$ (this particular one arises for the choice $X = A_1$). The vacuum moduli space should be described by the independent gauge-invariant chiral operators, modulo the F-term constraints. The gauge-invariant chiral mesonic operators are of form $A_i B_j C_k$, and after imposing the F-term conditions the ones that are left are ten in number:

$$A_1 B_1 C_1, A_1 B_1 C_2, A_1 B_1 C_3, A_2 B_2 C_2, A_1 B_2 C_2, A_2 B_2 C_3,$$
$$A_3 B_3 C_3, A_1 B_3 C_3, A_2 B_2 C_3, A_1 B_2 C_3. \tag{3.11}$$

But now, it can immediately be checked that these have a one-to-one correspondence with the ten monomials U_{ijk} that we introduced in defining the orbifold in section 2, and that the algebraic relations that they satisfy are precisely the same.[8] Therefore, the moduli space is precisely the orbifold $\mathbb{C}^3/\mathbb{Z}_3$.

4. Branes on the resolved orbifold

We saw in section 3 that the orbifold singularity of our orbifold can be "blown-up" by replacing it with the compact projective space $\mathbb{CP}^2$ so that one ends up with a non-compact Calabi-Yau manifold that looks asymptotically like $\mathbb{C}^3/\mathbb{Z}_3$. We will be interested in doing supergravity with D3-brane sources localized on this resolved space. In particular, we want the explicit form of the metric on it so that we can compute the Laplacian and solve for the warp factor in the full supergravity solution.

A strategy for the derivation of this metric (which generalizes readily to other $\mathbb{C}^n/\mathbb{Z}_n$ spaces [14, 15]) is given in an appendix. In terms of the angular variables defined by

$$w_1 = r \sin \sigma \sin \frac{\theta}{2}\, e^{i\left(\frac{\psi}{3} + \frac{\phi}{2} - \frac{\beta}{2}\right)}, \tag{4.1}$$

$$w_2 = r \sin \sigma \cos \frac{\theta}{2}\, e^{i\left(\frac{\psi}{3} - \frac{\phi}{2} - \frac{\beta}{2}\right)}, \tag{4.2}$$

$$w_3 = r \cos \sigma\, e^{i\frac{\psi}{3}}, \tag{4.3}$$

the metric takes the form (with ρ defined by $\rho^6 = r^6 + a^6$),

$$ds^2 = \frac{d\rho^2}{\left(1 - \frac{a^6}{\rho^6}\right)} + \frac{\rho^2}{9}\left(1 - \frac{a^6}{\rho^6}\right)\left(d\psi - \frac{3}{2}\sin^2\sigma(d\beta + \cos\theta d\phi)\right)^2 +$$
$$+ \rho^2\left(d\sigma^2 + \frac{1}{4}\sin^2\sigma(d\theta^2 + \sin^2\theta d\phi^2) + \frac{1}{4}\sin^2\sigma\cos^2\sigma(d\beta + \cos\theta d\phi)^2\right). \tag{4.4}$$

---

[8]Roughly speaking we find that the F-terms identify z_i with the fields (A_i, B_i, C_i).

In what follows, we will move back and forth freely between the radial coordinates r and ρ. The form of the metric demonstrates that far away from the resolution ($\rho \to \infty$), the angular part of the metric reduces to that of the Lens space $S^5/\mathbb{Z}_3$ thought of as a U(1) fibration over $\mathbb{CP}^2$. The second term on the first line in (4.4) corresponds to the U(1) fibration, and the second line corresponds to the $\mathbb{CP}^2$ base. The $\mathbb{CP}^2$ metric here is the standard Fubini-Study metric (see for instance, equation (5.2) in [16]). It should be emphasized that the ranges of the angles involved are important here. To get $\mathbb{CP}^2$ as the base, we need the periodicity of β to be 4π and the ranges of ϕ and θ to be 2π and π respectively. To make sure the fibration is indeed the Lens space, which is what corresponds to the resolution of the orbifold, we also need to fix the ψ-periodicity to be 2π. If instead we took the periodicity of ψ to be 6π, we get S^5 instead of $S^5/\mathbb{Z}_3$. To fix the range of σ we need to look at the defining relations for the angles in terms of the w_i given above. This fixes[9] the range of σ to be from 0 to $\pi/2$. For later use, we also write down $\sqrt{g_6}$ for the resolved $\mathbb{C}_3/\mathbb{Z}_3$,

$$\sqrt{g_6} = \frac{1}{24}\rho^5 \sin^3\sigma \cos\sigma \sin\theta, \qquad (4.5)$$

which happens to be the identical to the unresolved case in the $a \to 0$ limit.

The metric is clearly a higher dimensional generalization of the usual Eguchi-Hanson ALE gravitational instanton familiar from four dimensional gravity. The 4D Eguchi-Hanson is the analogous metric on the the resolution of $\mathbb{C}^2/\mathbb{Z}_2$.

Before proceeding let us stress again that there is a single parameter a, which controls the size of the "blown-up" cycle. This parameter corresponds to a dimension six operator on the gauge theory side. This follows from the standard AdS/CFT analysis exactly like in the case studied in detail in [39], where the authors considered a four cycle resolution of the $\mathbb{Z}_2$-orbifolded conifold.[10]

With this metric, our next task is to calculate the scalar Laplacian so that we can do supergravity as sketched in the last section. This is most easily done in terms of differential forms, using the fact that $\Box = *d*d$ where $*$ stands for the Hodge dual. The form of the metric immediately lets us write it as a sum of squares of (non-coordinate) basis forms, and working with them simplifies the computation of the Laplacian. This is because the Hodge duals are trivial to compute in terms of these, as opposed to the coordinate basis. The final result, after the dust settles, is

$$\Box H = \Box_\rho H + \frac{1}{\rho^2}\Box_\sigma H = \left(1 - \frac{a^6}{\rho^6}\right)^{1/2}\partial_\rho\left(\left(1 - \frac{a^6}{\rho^6}\right)^{1/2}\partial_\rho H\right) +$$
$$+ \frac{1}{\rho}\left(5 - \frac{2a^6}{\rho^6}\right)\partial_\rho H + \frac{1}{\rho^2}\left(\partial_\sigma^2 H + 2(\cot\sigma + \cot 2\sigma)\partial_\sigma H\right). \qquad (4.6)$$

---

[9]Another way to fix this is to use the fact that when the period of ψ is 6π we need to recover the 5-sphere, whose angle integral we know should be π^3. Using this and the angular part of the measure (4.5), we can integrate to find the σ-range.

[10]Using the terminology of [39] the a parameter corresponds to a "local" (dimension six) deformation, while the "global" (dimension two) deformation does not exist in our case.

The Laplacian here is written under the assumption that $H = H(\rho, \sigma)$, so the other angular coordinates drop off. But we do present the full Laplacian in an appendix for the viewing pleasure of the reader.

The reason we can get away with looking at only the dependence on one of the angles, σ, is as follows. First, we are placing the stack of D3-branes precisely *at* the resolution $\rho = a$, where the U(1) fibration has shrunk to zero size and therefore the dependence can only be on ρ and the coordinates of the blown-up four-cycle. This is easily understood through an analogy in two dimensions. If we keep the source at any point other than the origin, we expect the Green's function to depend both on the radial coordinate and the polar angle. But if we choose to place the source precisely at the origin (where the one-cycle, the circle corresponding to the polar angle, has shrunk to zero size), then the Green's function is independent of the angle and is purely a function of the radius. The second simplification happens because by symmetry, we are free to place the stack on the "North pole" of the four-cycle, $\sigma = 0$. Once we make that choice, the same kind of argument as above applies again, and we have a solution that is independent of the rest of the angles of $\mathbb{CP}^2$. In any event, the end result is that we can capture all the interesting physics by just looking at the ρ, σ-dependence of the warp factor, H.

If we ignore the σ-dependence as well, on the other hand, we loose some physics, because then we are making the assumption that the D-brane sources are smeared over the resolved $\mathbb{CP}^2$ instead of being localized. A solution of this form, but for the case of the resolved conifold, was constructed by Pando Zayas and Tseytlin [13]. For the resolved $\mathbb{C}^3/\mathbb{Z}_3$ we present a smeared solution in appendix D. This gives us a nice consistency check: the full Green's function that we construct in the following should reproduce the smeared result, when we look at its singlet-under-σ part.

The equation that we need to solve for the unsmeared case takes the form:

$$\Box H = \Box_\rho H + \frac{1}{\rho^2}\Box_\sigma H = -\frac{3C}{4\pi^3\rho^5\sin^3\sigma\cos\sigma}\delta(\rho - a)\delta(\sigma). \tag{4.7}$$

The normalization on the delta function is again fixed by integrating over the suppressed angles. For future convenience, we will define $\sqrt{g_\rho} \equiv \rho^5$ and $\sqrt{g_\sigma} \equiv \sin^3\sigma\cos\sigma$. The standard technology for solving such Poisson-type equations dictates that we proceed by first solving $\Box_\sigma Y_l = -E_l Y_l$, where the Y_l satisfy

$$\int Y_l^*(\sigma)Y_{l'}(\sigma)\sqrt{g_\sigma}d\sigma = \delta_{ll'}, \tag{4.8}$$

$$\sum_l Y_l^*(\sigma_0)Y_l(\sigma) = \frac{1}{\sqrt{g_\sigma}}\delta(\sigma - \sigma_0). \tag{4.9}$$

The next step is to solve the remaining (radial) part,

$$\Box_\rho H_l(\rho, \rho_0) - \frac{E_l}{\rho^2}H_l(\rho, \rho_0) = -\frac{3C}{4\pi^3\rho^5}\delta(\rho - \rho_0). \tag{4.10}$$

while using the boundary conditions relevant to the problem. Then, it is easy to check that

$$H(\rho, \rho_0 = a, \sigma, \sigma_0 = 0) = \sum_l H_l(\rho, \rho_0 = a)Y_l^*(\sigma_0 = 0)Y_l(\sigma), \tag{4.11}$$

is the desired solution to (4.7).

The first step in this program is the solution of the angular part. This can in fact be recast as a hypergeometric equation, with the solution

$$Y_l = A_l \, {}_2F_1(-l, l+2, 1, \cos^2 \sigma) = A_l \, P_l^{(0,1)}(-\cos 2\sigma), \tag{4.12}$$

where we re-write the hypergeometric function in terms of an orthogonal (Jacobi) polynomial to emphasis the fact that they satisfy an orthonormality relation. The orthonormality relation of the Jacobi polynomials [25] in the present context takes precisely the form we want, i.e. eqn(4.8), provided we take the normalization to be

$$A_l = 2\sqrt{l+1}. \tag{4.13}$$

Notice also that with the malice of hindsight, we have written $E_l = 2l(2l+4)$. We mention also that the Jacobi polynomials are defined only if we choose l to be an integer. Since the angular "energy" eigen values of the harmonics on the d-sphere take the form $l(l+d-1)$ (with $d = 5$ for our case) for any integer l, we are missing half of the harmonics in our description. Let us now show that this is just an immediate result of the fact that we are interested only in the ψ-independent harmonics, since, as we have already explained, this angle collapses at the tip leaving only the $\mathbb{CP}^2$. Obviousely, any S^5 harmonics can be multiplied by the $r^{l(l+4)}$ factor making it a solution of the full $\mathbb{R}_6$ Laplacian equation, which we will denote by Ψ_l. On the other hand, any such solution can be expressed in terms of the $\mathbb{C}_3$ holomorphic coordinates (4.1). Morever, this dependence on w_i's have to be homogenous, both due to the r dependence of w_i's and because by definition Ψ_l is a regular function of the angles. This means that Ψ_l looks like a sum of terms each being a product of exactly l w_i's or $\bar{w}_i$'s. Finally, to cancel the ψ-dependence of Ψ_l we need equal number of w_i's and of $\bar{w}_i$'s in each term, so, indeed, only harmonics with even l can contribute to our expansion.[11]

In any event, with the angular harmonics at hand, we turn to the radial part which takes the form

$$\left(1 - \frac{a^6}{\rho^6}\right)\frac{d^2 H_l}{d\rho^2} + \left(\frac{5}{\rho} + \frac{a^6}{\rho^7}\right)\frac{dH_l}{d\rho} - \frac{2l(2l+4)}{\rho^2}H_l = -\frac{3C}{4\pi^3\rho^5}\delta(\rho - a). \tag{4.14}$$

Away from the stack, these can again be solved in terms of hypergeometric functions, and the independent solutions are

$$H_l^A = {}_2F_1\left(\frac{2}{3} + \frac{l}{3}, \ -\frac{l}{3}, \ \frac{2}{3}; \ \frac{\rho^6}{a^6}\right), \tag{4.15}$$

$$H_l^B = \frac{\rho^2}{a^2} \, {}_2F_1\left(1 + \frac{l}{3}, \ \frac{1}{3} - \frac{l}{3}; \ \frac{4}{3}; \ \frac{\rho^6}{a^6}\right). \tag{4.16}$$

The radial equation has a symmetry under $l \leftrightarrow -(l+2)$, which manifests itself in the solutions above as the symmetry of the hypergeometric function ${}_2F_1(a, \ b; \ c; z)$ under $a \leftrightarrow b$. This choice can be judiciously used to simplify some of the computations below.

---

[11]We are grateful to the anonymous referee for this elegant explanation.

We want to look for a specific solution that is defined for all $r > 0$ ($\rho > a$), with the condition that it should vanish at infinity. This fixes it upto an overall constant which in turn we can determine by integrating across the delta function. The word *across* here should be taken with a grain of salt because this is a radial delta function localized at the (equivalent of the) origin. Effectively this means that when we integrate (4.14), the entire contribution to the discontinuity on the first derivative comes from the "outside" piece because there is no inside piece at the origin.[12] The end result is:

$$H_l^{\text{out}} \equiv C_l H_l^O = C_l \left[H_l^A - \frac{\Gamma(1 + \frac{l}{3})^2 \Gamma(-\frac{1}{3})}{\Gamma(2 + \frac{l}{3})^2 \Gamma(\frac{1}{3})} H_l^B \right], \tag{4.17}$$

$$\text{where} \quad C_l = \frac{C \Gamma(-\frac{l}{3}) \Gamma(\frac{l}{3} + \frac{2}{3})}{4\pi^3 a^4 (3 - \sqrt{3} \cot(\frac{\pi l}{3})) \Gamma(\frac{2}{3})}. \tag{4.18}$$

In fixing the normalization above, it is useful to notice that the unnormalized warp factor H_l^O behaves near $r = 0$ as

$$H_l^O \longrightarrow \frac{3(\sqrt{3} \cot(\frac{\pi l}{3}) - 3) \Gamma(\frac{2}{3})}{\Gamma(\frac{-l}{3}) \Gamma(\frac{l}{3} + \frac{2}{3})} \log(r), \tag{4.19}$$

as can be checked.

Using all the above ingredients, and using (4.11), we can finally write down the warp factor for the stack on the "North pole" of the resolution as

$$H(\rho, \sigma) = \sum_{l=0}^{\infty} 4(-1)^l (l+1)^2 P_l^{(0,1)}(-\cos 2\sigma) \; H_l^{\text{out}}(\rho). \tag{4.20}$$

We have used the fact that $P_l^{(0,1)}(-1) = (-1)^l (l+1)$.

4.1 Consistency checks and holographic RG flow

As already commented, the solution found above should reproduce the result of appendix D, when we restrict ourselves to the $l = 0$ terms. This is indeed what we find:

$$H(\rho, \sigma)|_{(l=0)} = 4 H_{l=0}^{\text{out}}(\rho) = \frac{16\pi^2 g_s N \alpha'^2}{a^4 \sqrt{3}} \left[1 + \frac{(a^6 + r^6)^{1/3}}{a^2 \Gamma(\frac{4}{3}) \Gamma(\frac{2}{3})} \, _2F_1\left(\frac{1}{3}, 1; \frac{4}{3}; 1 + \frac{r^6}{a^6}\right) \right]$$

$$= \frac{12\pi g_s N \alpha'^2}{r^4} \, _2F_1\left(\frac{2}{3}, \frac{2}{3}; \frac{5}{3}; -\frac{a^6}{r^6}\right), \tag{4.21}$$

where, in the last line we have used a functional identity relating hypergeometric functions [26].[13] From the asymptotic of the hypergeometric function, it is clear that at $r \to \infty$ the correction from the $\frac{1}{r^4}$ behavior is proportional to a^6, which ties in with the expectations from [39].

---

[12]To be more precise, we should put the delta function away from $r_0 = 0$ ($\rho_0 = a$) and then let $r_0 \to 0$ *after* the matching, but for sufficiently well-behaved functions, this gives the same result as our recipe here.

[13]Incidentally, the same identity can be used to rewrite (4.17) as $\sim \frac{1}{r^{4+2l}} \, _2F_1\left(\frac{2+l}{3}, \frac{2+l}{3}; \frac{5+2l}{3}; -\frac{a^6}{r^6}\right)$, which has the advantage that its convergence at infinity is manifest.

It can also be checked that this singularity at $r = 0$ arising from the smearing is removed by the sum over the various l's. In our case, from the small r behavior of H_l^{out} presented above, we see that this sum takes the form

$$
\begin{aligned}
H &= \frac{-C}{8\pi^3 a^4} \log(\rho^6 - a^6) \sum_{l=0}^{\infty} 4(-1)^l (l+1)^2 P_l^{(0,1)}(-\cos 2\sigma) \\
&= \frac{-C}{8\pi^3 a^4} \log(\rho^6 - a^6) \frac{\delta(\sigma)}{\sqrt{g_\sigma}},
\end{aligned}
\tag{4.22}
$$

where in the last line, we have used the completeness relation (4.9) for Jacobi polynomials [27]. This makes it immediately clear that the singularity of in the smeared case at $r = 0$ is removed because of the vanishing of the delta-function away from the location of the stack ($\sigma = 0$), in our case. The smearing of the source on the four-cycle is also evident because the radial part takes the form $\log(\rho^6 - a^6) \sim \log r$, which is nothing but the Green's function in (the remaining) two dimensions.

We can in fact do more, if we keep track of the l-dependence of the H_l^{out} in the sum. In fact, the sum of all the various l pieces near $r = 0$ should gives rise to an AdS throat, because around a smooth point, all spaces are locally flat. The emergence of the throat is most easily seen if we approach $r = 0$ along $\sigma = 0$, because then the warp factor looks like

$$
H(r) = \sum_{l=0}^{\infty} 4(l+1)^3 H_l^{\text{out}}(r).
\tag{4.23}
$$

We are interested in the near-horizon behavior where the local curvatures have become negligible, which means we are working in the limit where the distance scales are much less than the resolution size, $r \ll a$. We can solve the radial equation (away from the source) in this limit, and the solution that dies down at infinity can be expressed in terms of modified Bessel functions of the second kind [28]. The details of the solution are irrelevant to us, except for one piece of information: the entire dependence of the solution on r and l (the normalization fixed by integrating across the delta function turns out to be independent of l), is captured by the combination $\sqrt{l(l+2)}\, r^3 \sim l\, r^3$. So we can write

$$
H(r) \sim \sum_{l}^{\infty} l^3 f(lr^3),
\tag{4.24}
$$

where the fact that we expect this sum to be convergent in l implies that the function $f(lr^3)$ can be thought of as a regulator.[14] In particular, the regulator accomplishes finiteness by decaying rapidly for $l > \frac{1}{r^3}$, so we can restrict the sum as

$$
H(r) \sim \sum_{l=0}^{1/r^3} l^3 f(lr^3) \sim \int_0^{1/r^3} l^3 f(lr^3) dl \sim \frac{1}{r^{12}} \int_0^1 x^3 f(x) dx = \frac{\text{const.}}{r^{12}}.
\tag{4.25}
$$

The second to last step involves a change of variables, and it can be checked that the final integral converges for the modified Bessel function mentioned earlier: for small r, it can be

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[14]This clever trick is taken straight from Klebanov-Murugan [12].

approximated by $\log r$. All that remains, is to notice that close to $\rho = a$ (with $\sigma = 0$), the metric (4.4) takes the flat form with a new radial coordinate $u = \frac{r^3}{3a^3}$. So in terms of this flat coordinate, the warp factor goes as $\sim \frac{1}{u^4}$, suggesting the emergence of the AdS throat through the usual arguments.

5. RG flow in the gauge theory

In this section we will describe various RG flows triggered by non-zero VEVs of the fields A_i, B_i and C_i.

We will start with the unresolved case. The supergravity counterpart of this discussion can be found in appendix C. The unresolved case corresponds to $t_a = 0$ in (3.7). As we have already explained above the set (3.7) does not describe the moduli space, since we also have to impose the F-terms conditions. For instance, it immediately follows that setting $A_1 = B_2 = C_3 = v$ and giving zero VEVs to the rest of the fields is consistent with the D-terms equations, but still does no satisfy the F-term restrictions, since $A_1 B_2 C_3 \neq A_2 B_3 C_1 = 0$. The VEVs that do not contradict the F-term relations are of the form $A_3 = B_3 = C_3 = v$. We see that in this case $U_{333} = v^3$ and all the other mesons are zero, so the VEVs corresponds to a point on the unresolved orbifold away from the apex. To be more precise, comparing (2.10) and (3.11) we learn that $|z_3| = \frac{1}{\sqrt{3}}|z_4| = v^{3/4}$.

Let us now analyse the superpotential. The $SU(N)_1 \times SU(N)_2 \times SU(N)_3$ gauge group is broken down to a single $SU(N)$. Substituting $A_3 = B_3 = C_3 = v$ into (3.10) we find that the mass matrix for the remaining six fields has rank four.[15] Being more specific, the eigenvalues are 0, $\frac{\sqrt{3}}{2}v$ and $-\frac{\sqrt{3}}{2}v$, each having degeneracy two. The matrix appears in appendix E, where we also report its related eigenvectors. Since out of three $SU(N)$ gauge groups two are broken by the VEVs, two chiral superfields should be "eaten" by the corresponding vector multiplets to form two massive multiplets. This is easily seen in the unitary gauge. Indeed, the gauge is completely fixed by putting $A_3 = B_3 = v$, while we still have to take into account the fluctuations of C_3. As for the rest of the fields the following parameterization proves to be convenient:

$$C_3 = C_3 + \Phi, \qquad A_1 = \Phi_1 + 2\Psi_1, \qquad B_1 = \Phi_1 - \Psi_1 - \Psi_4, \quad C_1 = \Phi_1 - \Psi_1 + \Psi_4,$$
$$A_2 = \Phi_2 + 2\Psi_3, \quad B_2 = \Phi_2 + \Psi_2 - \Psi_3, \quad C_2 = \Phi_2 - \Psi_2 - \Psi_3. \tag{5.1}$$

Substituting this into the superpotential we find:

$$W = 6v\,\mathrm{Tr}(\Psi_1\Psi_2 + \Psi_3\Psi_4) + \mathrm{Tr}\left(\Phi\left[\Phi_1, \Phi_2\right]\right) -$$
$$-\mathrm{Tr}\bigg(\Phi\Big(\{\Phi_1, \Psi_2\} + \{\Phi_2, \Psi_4\} + [\Phi_1, \Psi_3] - [\Phi_2, \Psi_1] +$$
$$+\{\Psi_1, \Psi_2\} + \{\Psi_3, \Psi_4\} + [\Psi_1, \Psi_3] - [\Psi_2, \Psi_4]\Big)\bigg). \tag{5.2}$$

As expected four fields, Ψ_1, Ψ_2, Ψ_3, and Ψ_4, become massive and we have to integrate all of them out. As a result all the terms in the last two lines of the superpotential expression

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[15]We would like to thank the referee for pointing out to us an obvious mistake regarding the matrix rank in the earlier version of the paper.

become quartic in terms of the three surviving fields Φ, Φ_1 and Φ_2. The quartic terms will be irrelevant in the IR, thus we are left only with the second term of (5.2), which is precisely the $\mathcal{N} = 4$ superpotential.[16]

We now consider the resolved orbifold case. For D3-branes localized away from the resolved apex, the RG flow is essentially the same as for the unresolved case. The only difference is that the VEVs of A_3, B_3 and C_3 cannot anymore be equal. Let us focus instead on D3-branes located at the point $|z_3|^2 = \mu$ of $\mathbb{CP}^2$.

In this case, we found that there are two different scenarios that describe the RG flow. According to the first, two bi-fundamentals acquire VEVs (say $A_3 = B_3 = v$), while in the second only one VEV is non-trivial (without losing generality we will put $C_3 = v$). Notice that in both cases all the mesonic fields vanish and the SU(3) symmetry is broken down to SU(2) matching the geometry expectations. Unfortunately, since we don't know the precise form of the dual dimension six operator, we cannot directly check what scenario is the right one. Instead, we will describe both possibilities arguing that in any case the RG flow re-produces the $\mathcal{N} = 4$ field content and the superpotential.

5.1 The $A_3 = B_3 = v$ RG flow

Again, these VEVs leave only one un-broken SU(N). Furthermore, plugging the VEVs into the superpotential we find that the fields C_1 and C_2 become massive, so we have to integrate them out. This, in turn, leads to two constraints: $A_1 = B_1$ and $A_2 = B_2$. Substituting this into the remaining two terms in the superpotential we immediately arrive at the $\mathcal{N} = 4$ superpotential:

$$W = \mathrm{Tr}\left(C_3\left[A_1, A_2\right]\right). \tag{5.3}$$

5.2 The $C_3 = v$ RG flow

Now the superpotential reads:[17]

$$W = vA_1B_2 - vA_2B_1 + A_2B_3C_1 - A_1B_3C_2 + A_3B_1C_2 - A_3B_2C_1 \tag{5.4}$$

Integrating the massive fields $A_{1,2}$ and $B_{1,2}$ out we arrive at the following result:

$$W = \frac{1}{v}\mathrm{Tr}\left(A_3B_3\left[C_1, C_2\right]\right). \tag{5.5}$$

The VEV of C_3 breaks also SU(N)$_1$ × SU(N)$_2$ down to the diagonal SU(N) so we end up with the quiver depicted in figure (3). This theory, however, is not conformal anymore, since we now have $N_{\mathrm{f}} = N_{\mathrm{c}}$ for the 3d node on the quiver on figure (3). The theory flows to the strongly coupled region and the right node confines. Apart from the adjoint fields C_1 and C_2 we have a meson $M = A_3B_3$, which also transforms in the adjoint of the left node of the quiver diagram. The field content, therefore, exactly reproduces the

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[16]Alternatively one can notice that the fields Ψ_i have dimension two, so all the operators in the last two lines of (5.2) are IR irrelevant.

[17]For the sake of simplicity we omit here the group traces and indices.
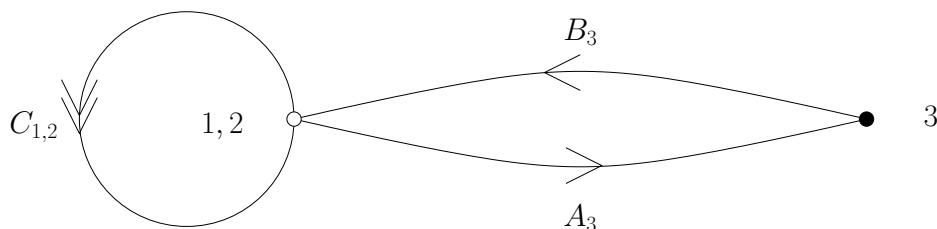
Figure 3: The gauge theory quiver for the theory after the first "step" of the RG flow. The VEV of the field C_3 breaks $SU(N)_1 \times SU(N)_2$ down to $SU(N)_{1,2}$ and the massive fields $A_{1,2}$ and $B_{1,2}$ are integrated out. At the second step, node 3 confines.

$\mathcal{N} = 4$ SYM theory. The superpotential, however, involves an additional non-perturbative Affleck-Dine-Seiberg term:

$$W = \frac{1}{v}\text{Tr}\left(M\left[C_1, C_2\right]\right) + X(\det M - B\tilde{B} - \Lambda^{2N}). \tag{5.6}$$

Here B and $\tilde{B}$ are the baryon fields, Λ is the strong coupling scale and the chiral field X is the Lagrange multiplier imposing the quantum moduli constraint in the parentheses. In the deformed conifold model this constraint exhibits two completely *separated* branches of solutions [31, 32], since the meson and the baryons cannot acquire VEVs simultaneousely. In the present model the two branches are not separated. This, however, does not modify the observation that the superpotential is still cubic. For instance, the "purely" mesonic branch corresponds to $B = \tilde{B} = 0$ and $\det M = \Lambda^{2N}$, which can solved by $M = \mathbf{1}_{N \times N}\Lambda^2$. Substituting this into (5.6) we find that the superpotential vanishes, so, similar to what we did in the unresolved case, we have to consider next to the leading order corrections. We will set:

$$M = \mathbf{1}_{N \times N}\Lambda^2 + \frac{\Phi}{v}, \tag{5.7}$$

reproducing eventually the $\mathcal{N} = 4$ superpotential:

$$W = \text{Tr}\left(\Phi\left[C_1, C_2\right]\right). \tag{5.8}$$

6. Discussion

In this paper we have constructed a supergravity solution that describes an RG flow from the gauge theory dual to $AdS_5 \times S^5/\mathbb{Z}_3$, to $\mathcal{N} = 4$ SYM theory which in turn is dual to

$AdS_5 \times S^5$. The resolution of the $\mathbb{Z}_3$ singularity is given by a $\mathbb{CP}^2$ placed at the tip of the cone. From the explicit metric that we wrote down, it can be checked [39] that far away in the UV, the leading order correction from AdS describes an order six operator acquiring VEV in the gauge theory. This VEV is nothing but the resolution parameter in the geometry. The explicit form of this operator unfortunately is unknown.

The problem of constructing a 10d supergravity background based on the resolved 6d space reduces to finding the six-dimensional Green function, which serves as a warp function in the full solution. If the function, however, is taken to depend only on the radial coordinate, and so the D3-brane source is "smeared" over the $\mathbb{CP}^2$, the total 10d solution proves to be singular. This is just a common phenomenon for 10d backrounds based on "smeared" D-branes.[18] To avoid the singularity we have placed the D3 branes stuck at a point of the $\mathbb{CP}^2$. The map between the D3 branes coordinates and the gauge invariant (mesonic) combination of the fields was presented in section 3.

Back in the dual gauge theory, this corresponds to giving VEVs to some bi-fundamental fields. Although the precise form of the dimension six operator dual to the resolution parameter remains a mystery, we are able to show that there are only two distinct ways in which the fields can acquire non-trivial VEVs. In both cases the non-trivial VEVs break the SU(3) isometry, the conformal invariance of the original theory and trigger (presumably) different RG flows to the same $\mathcal{N} = 4$ SYM theory.

We have argued that in the second scenario the RG flow proceeds in two "steps". First, some fields acquire VEVs breaking three gauge groups down to two. Integrating the massive fields out, we are left with four (two adjoints and two bi-fundamentals) out of nine original fields and a *quartic* superpotential. Second, one of the nodes confines. In terms of the two adjoint fields and a new meson constructed from the two bi-fundamentals, the superpotential becomes *cubic* like in the $\mathcal{N} = 4$ SYM theory. It is very tempting, therefore, to identify these two steps on the supergravity side. Unfortunately, it does not look possible if one wants to stick with the supergravity approximation, which requires large t'Hooft coupling $\lambda = Ng_{YM}^2$. Indeed, the strong coupling scale on the second "step" of the RG flow is related to the energy scale E set by the VEV v as:

$$\Lambda = E e^{-\frac{8\pi^2}{2Ng_{YM}^2(E)}} \qquad \text{where} \qquad E = \sqrt{v}. \tag{6.1}$$

We see that, for large $\lambda = Ng_{YM}^2$ the scale Λ and E are of the same order and we cannot distinguish between them.[19] This situation is actually familiar from the conifold model cascade [32], where on the supergravity side the cascade steps are "smoothed" out for the same reasons as in our case.

A number of interesting questions can be raised for further investigation. In [12] the presence of the baryonic VEV was verified from the analysis of the DBI action of an Euclidian D3-brane wrapped on a 4d cycle of the resolved conifold. Similar computation can also be performed in our case. Remarkably, from the gauge theory point of view this condensate has to be the same in both suggested RG flow scenarios. Indeed, the baryon

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[18]See [13] for the resolved conifold example.

[19]We are grateful to Riccardo Arguio for explaining this subtlety.

constructed from the field A_3B_3 and the baryon built from C_3 describe the same baryonic operator, since their product can be written only in terms of the meson $A_3B_3C_3$. This, and the fact that the Fayet-Iliopoulos parameters (3.7)–(3.9) turn out to be identical for both these choices, leads us to speculate that perhaps the two RG flows are dual descriptions of each other.

Finally, it will be very interesting to find the dimension six operator dual to the resolution parameter a. We hope that our paper might provide useful results towards this direction, along the lines of [39].

Acknowledgments

A. Metric on the ALE space

One way to derive the metric on the resolution of $\mathbb{C}^3/\mathbb{Z}_3$ is to use the fact that the Kähler form $K(r)$ on this space depends only on $r^2 = |w_1|^2 + |w_2|^2 + |w_3|^2$. Since the space is Calabi-Yau, among other things, it is both Rici-flat and Kähler. So the metric can be written as $g_{i\bar{j}} = \partial_i \partial_{\bar{j}} K$, and then the Ricci-flatness condition turns out to be a differential equation for $K(r)$:

$$\det(\partial_i \partial_{\bar{j}} K) = \text{const.} \tag{A.1}$$

After absorbing the irrelevant constant by rescaling $K(r)$, this translates to

$$(K')^2 (r^2 K'' + K') = 1. \tag{A.2}$$

The primes here are with respect to r^2. It will prove convenient to introduce a new function $\mathcal{F}$ defined by

$$\mathcal{F} \equiv r^2 K', \tag{A.3}$$

in terms of which the differential equation above has the simple solution

$$\mathcal{F} = (r^6 + a^6)^{1/3}, \tag{A.4}$$

with a^6 an integration constant which reflects the resolution of the space. By tuning a to zero, we can recover the unresolved space. It is easy to integrate $\mathcal{F}$ once again to obtain an explicit expression for $K(r)$ in terms of hypergeometric functions, but we will not need it, so we will refrain from writing it down.

Using these, and the angular variables defined in section 4 in terms of the w_i, we can calculate the metric directly as $ds^2 = g_{i\bar{j}} dw^i d\bar{w}^{\bar{j}}$. The result is

$$ds^2 = \mathcal{F}' dr^2 + \frac{\mathcal{F}' r^2}{9} \Big(d\psi - \frac{3}{2} \sin^2 \sigma (d\beta + \cos\theta d\phi) \Big)^2 +$$
$$+ \mathcal{F}\Big(d\sigma^2 + \frac{1}{4} \sin^2 \sigma (d\theta^2 + \sin^2 \theta d\phi^2) + \frac{1}{4} \sin^2 \sigma \cos^2 \sigma (d\beta + \cos\theta d\phi)^2 \Big). \quad \text{(A.5)}$$

This turns into the metric presented in section 4, once we make the definition $\rho^2 \equiv \mathcal{F}$.

B. The scalar Laplacian

In this appendix we exhibit the full Laplacian for the resolved $\mathbb{C}^3/\mathbb{Z}_3$ space. It takes the form

$$\Box H = \Big(1 - \frac{a^6}{\rho^6} \Big)^{1/2} \partial_\rho \Big(\Big(1 - \frac{a^6}{\rho^6} \Big)^{1/2} \partial_\rho H \Big) + \frac{1}{\rho} \Big(5 - \frac{2a^6}{\rho^6} \Big) \partial_\rho H +$$
$$+ \frac{1}{\rho^2} \Big[(\partial_\sigma^2 H + 2(\cot\sigma + \cot 2\sigma)\partial_\sigma H) + \frac{4}{\sin^2 \sigma \sin\theta} \partial_\theta(\sin\theta \partial_\theta H) +$$
$$+ \frac{4}{\sin^2 \sigma} \Big(\frac{1}{\sin\theta} \partial_\phi - \cot\theta \partial_\beta \Big)^2 H + \frac{4}{\cos^2 \sigma} \Big(\frac{3}{2} \sin\sigma \partial_\psi + \frac{1}{\sin\sigma} \partial_\beta \Big)^2 H +$$
$$+ 9 \Big(1 - \frac{a^6}{\rho^6} \Big)^{-1} \partial_\psi^2 H \Big]. \quad \text{(B.1)}$$

C. Branes on the unresolved $\mathbb{C}^3/\mathbb{Z}_3$: away from the tip

Here we compute the Green's function for a D-brane stack on the unresolved orbifold, but away from the tip ($\rho \neq a$). Far away from the stack, we expect to reproduce the behavior we calculated in section 3.1, but we also expect to see the AdS throat in the near-horizon region because the stack is no longer at a singular point. The fact that the space is unresolved will be reflected in the fact that the solution is still singular.

To simplify the computations, we will look at the case where the stack is at the point $\rho = \rho_0$, $\sigma = 0$. The location $\sigma = 0$ kills the dependence of the Green function on the other angles of the $\mathbb{CP}^2$ as was argued in the main body of the paper. We will only be interested in the case where the solution is of the form $H(\rho, \sigma)$, which means that we are assuming that the D3-branes are smeared over the ψ-circle.[20] The form of the Laplacian permits such a choice. The equation to be solved takes the form

$$\Box_\rho H + \frac{1}{\rho^2} \Box_\sigma H = \frac{-3C}{4\pi^3 \rho^5 \sin^3 \sigma \cos\sigma} \delta(\rho - \rho_0) \delta(\sigma), \quad \text{(C.1)}$$
$$\text{where} \quad \Box_\rho = \frac{1}{\rho^5} \frac{\partial}{\partial\rho} \Big(\rho^5 \frac{\partial}{\partial\rho} \Big),$$

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[20]It turns out that the more general case of ψ-dependent warp factor can also be solved exactly in terms of Jacobi polynomials (this time, with more general quantum numbers than what we saw in section 4), but we will not present the solution here.

with $\Box_\sigma$ as defined by (4.6). As before, we now solve

$$\left(\Box_\sigma + 2l(2l+4)\right)\Sigma(\sigma) = 0, \tag{C.2}$$

for the angular part in terms of Jacobi polynomials. The full normalized angular solutions are

$$Y_l(\sigma) = \sqrt{4(l+1)}P_l^{(0,1)}(-\cos 2\sigma). \tag{C.3}$$

The remaining radial part of the equation looks like

$$\frac{\partial^2 H_l}{\partial\rho^2} + \frac{5}{\rho}\frac{\partial H_l}{\partial\rho} - \frac{2l(2l+4)}{\rho^2}H_l = \frac{-3C}{4\pi^3\rho^5}\delta(\rho - \rho_0), \tag{C.4}$$

whose solution, after matching the function and its derivative through the delta function is

$$H_l(\rho, \rho_0) = \begin{cases} \dfrac{3C}{16\pi^3(l+1)}\rho_0^{-(2l+4)}\rho^{2l} & \rho \le \rho_0, \\[3mm] \dfrac{3C}{16\pi^3(l+1)}\rho^{-(2l+4)}\rho_0^{2l} & \rho \ge \rho_0. \end{cases} \tag{C.5}$$

The full solution looks like

$$H(\rho, \rho_0; \sigma, 0) = \sum_l H_l(\rho, \rho_0)Y_l^*(0)Y_l(\sigma). \tag{C.6}$$

Restricting to $l = 0$ gives the dependence far away from the source

$$H(\rho) \to \frac{3C}{16\pi^3\rho^4} \times \left(\sqrt{4}\,\right)^2 = \frac{12\pi g_s N\alpha'^2}{\rho^4} \tag{C.7}$$

which reproduces the result obtained in section 3.1, eq. (3.5), where the stack was assumed to be at the tip. The emergence of the AdS throat close to the stack is entirely analogous to the resolved case, which was discussed in the main text.

D. Warp factor with smeared sources

In this appendix, we compute the warp factor for the resolved $\mathbb{C}^3/\mathbb{Z}_3$, but with the assumption that the source branes are smeared all over the resolution and there is no angular dependence. We start with the Laplacian as defined by

$$\Box H = \frac{1}{\sqrt{g_6}}\,\partial_\mu(\sqrt{g_6}g^{\mu\nu}\partial_\nu H) = \frac{1}{\rho^5}\partial_\rho\left(\rho^5\left(1 - \frac{a^6}{\rho^6}\right)\partial_\rho H\right) \tag{D.1}$$

where we have used (4.5) and assumed $H \equiv H(\rho)$, which is the basis of the smeared source approximation. For comparison with the unsmeared solution in the main body of the paper, we will solve the Green's equation in terms of $r = (\rho^6 - a^6)^{1/6}$. The equation is effectively

first order and can be solved by direct integration in terms of elementary functions. But we will write its solution in a hypergeometric form

$$H = \frac{12\pi g_s N \alpha'^2}{r^4} \, _2F_1\left(\frac{2}{3}, \, \frac{2}{3}; \, \frac{5}{3}; -\frac{a^6}{r^6}\right), \tag{D.2}$$

for ease of comparison. We have fixed the overall normalization by demanding that the warp-function should look like that of the unresolved case (3.5), far away from the resolution ($\frac{a}{r} \to 0$).

E. The mass matrix for the unresolved case

We have argued above that the VEVs $A_3 = B_3 = C_3 = v$ render the superpotential quadratic in terms of the remaining fields and the symmetric mass matrix $\mathbf{M}$ has rank four. In this appendix we present the basis in which the mass matrix acquires a diagonal form. In the $(A_1, A_2, B_1, B_2, C_1, C_2)$ basis we have:

$$\mathbf{M} = \frac{v}{2} \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The matrix eigenvalues are 0, $\frac{\sqrt{3}}{2}v$ and $-\frac{\sqrt{3}}{2}v$. Each of the eigenvalues has degeneracy two. The corresponding eigenvectors are as follows:

$$\begin{pmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{3}} \\ -\frac{1}{2\sqrt{3}} \\ -\frac{1}{2\sqrt{3}} \\ 0 \\ \frac{1}{2} \\ -\frac{1}{2} \end{pmatrix} \begin{pmatrix} 0 \\ -\frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{\sqrt{3}} \\ -\frac{1}{2\sqrt{3}} \\ -\frac{1}{2\sqrt{3}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{3}} \\ -\frac{1}{2\sqrt{3}} \\ -\frac{1}{2\sqrt{3}} \\ 0 \\ -\frac{1}{2} \\ \frac{1}{2} \end{pmatrix} \begin{pmatrix} 0 \\ \frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{\sqrt{3}} \\ -\frac{1}{2\sqrt{3}} \\ -\frac{1}{2\sqrt{3}} \end{pmatrix} \tag{E.1}$$

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